The Small Angle Approximation

1 INTRODUCTION

A simple harmonic motion (SHM) describes systems that oscillate predictably with time. One system that exhibits SHM is an oscillating mass attached to a spring. Another system that approximately exhibits SHM is a simple pendulum.

When solving the equation of motion for a pendulum an assumption is typically made to simplify the math. The assumption – called the “small angle approximation” – limits the amplitude of the pendulum to small angles. With this assumption, the equations that describe the motion of a pendulum are identical to SHM. The equations of motion for a pendulum are often accompanied by the disclaimer that the solution is good for “sufficiently small angles”.

This lab explores the limitations of the small angle approximation in a simple pendulum.

2 LEARNING OBJECTIVES

At the conclusion of this activity you should be able to:

- Explain what is meant by the small angle approximation.
- Design an experiment to measure the angle at which the small angle approximation is no longer valid.

3 BACKGROUND

3.1 MATHEMATICS: PLOTTING A SINE FUNCTION

Sine and cosine functions are often used to model periodic systems[1][2]. A generalized sine wave can be written as:

\[ f(x) = A \sin(Bx + C) + D, \quad (3.1) \]

where the parameters \( A \), \( B \), \( C \), and \( D \) specify the shape and position of the function (see Figure 3.1). A detailed description of each of the parameters follows:

- **\( A \) = Amplitude**: The amplitude refers to the swing of the sine function. The amplitude is typically measured as the “peak amplitude” or the maximum absolute value of the function.

- **\( B \) = Angular Frequency**: The angular frequency describes the number of cycles of the function in a given time period. The period \( T \) – time required for a single oscillation – is related to the parameter \( B \) by:

\[ T = \frac{2\pi}{B}. \quad (3.2) \]
For a system changing in time, $B$ is typically expressed in radians per unit time and is often labeled $\omega$.

$C =$ **Phase**: The phase describes a horizontal translation of the signal. The phase shifts the signal a horizontal distance of $C/B$. Positive values of $C$ result in *negative* translation along the $x$ axis. The phase is often labeled $\phi$.

$D =$ **Offset**: The offset parameter translates the function in the vertical direction. A positive $D$ moves the signal in the positive $y$ direction.

$$f(x) = A \sin(Bx + C) + D$$

![Figure 3.1: A cartoon of one cycle of a sine function showing the effect of each of the parameters: $A$, $B$, $C$, and $D$ on the shape and position of the graph.](image)

3.2 **TAYLOR SERIES**

A Taylor series is a mathematical tool to represent a function near a point in terms of an infinite sum[3][4]. The Taylor series for sine and cosine are:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \quad (3.3)$$

and

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \quad (3.4)$$

The more terms that are included in a Taylor polynomial, the closer the series approximates the original function (see Figure 3.2).
In physics, we often estimate the solution for a complicated system by including only the first term (i.e. “first order”) of the Taylor series. In this case sine and cosine are approximated by:

\[
\sin(x) \approx x \quad \text{(3.5)}
\]

\[
\cos(x) \approx 1. \quad \text{(3.6)}
\]

For sufficiently small values of \( x \), this is often an acceptable approximation.

![Taylor Series for \( f(x) = \sin x \)](image)

Figure 3.2: The first three terms of the Taylor series for \( \sin x \). For small values of \( x \) it is clear that even the first order (red line) is a good approximation of the function.

### 3.3 The Motion of a Pendulum

One way to analyze the forces acting on a pendulum is in terms of torque[5]. Recall that torque \( \vec{\tau} \) is given by:

\[
\vec{\tau} = \vec{r} \times \vec{F} = I \vec{\alpha},
\]

where \( \vec{r} \) is the vector from the point of rotation to the force \( \vec{F} \), \( I \) is the moment of inertia, and \( \vec{\alpha} \) is the angular acceleration. Recall that the angular acceleration is the second time derivative of the displacement angle, \( \theta \). That is:

\[
\vec{\alpha} = \frac{d^2\theta}{dt^2}. 
\]

Consider a simple pendulum with a bob of mass \( m \) and a massless chord of length \( l \) as shown in Figure 3.3. The moment of inertia for this system is given by:

\[
I = ml^2. 
\]
The restoring force of the pendulum is given by:

\[ |\vec{F}_r| = -mg \sin \theta. \]  \hspace{1cm} (3.10)

Note that the restoring force is perpendicular to the chord and opposes the displacement.

Starting from Equation 3.7, with \( \vec{r} = l \) and substituting Equation 3.9 and 3.10 we obtain:

\[ (ml^2)\alpha = l \times (-mg \sin \theta) \]  \hspace{1cm} (3.11)

Simplifying and expressing \( \alpha \) in terms of the second time derivative of \( \theta \) (Equation 3.8).

\[ \frac{d^2 \theta}{dt^2} = -\frac{g}{l} \sin \theta. \]  \hspace{1cm} (3.12)

Obtaining an exact solution for Equation 3.12 is difficult. In order to solve the equation more easily, we make the assumption that the amplitude of the pendulum oscillations is be small – this is the small angle approximation[6]. Under this condition, we can use the first-order Taylor approximation described in Equation 3.5.

With this assumption, Equation 3.12 simplifies to equation:

\[ \frac{d^2 \theta}{dt^2} \approx -\frac{g}{l} \theta, \]  \hspace{1cm} (3.13)

We define the angular frequency \( \omega \) in terms of \( g \) and \( l \):

\[ \omega = \sqrt{g/l}. \]  \hspace{1cm} (3.14)
and re-write the equation of motion as:

\[
\frac{d^2\theta}{dt^2} \approx -\omega^2 \theta.
\] (3.15)

Equation 3.15 is the general form for simple harmonic motion. Masses oscillating on springs and simple pendulums are two examples of systems that exhibit simple harmonic motion.

The solutions to Equation 3.15 are given by:

\[
\theta(t) = A \sin(\omega t + \phi). 
\] (3.16)

For a pendulum, \(A\) is the amplitude, or maximum opening angle \(\theta_{max}\), that the pendulum makes with respect to vertical. \(\omega\) is the angular frequency as defined in Equation 3.14 and \(\phi\) is known as the phase angle. Note the similarities between Equation 3.16 and 3.1.

4 Procedure

To obtain Equation 3.15 we limit ourselves to small angles. What exactly qualifies as a small angle? The goal of this lab is to study the motion of a pendulum to determine when the small angle approximation breaks down.

For several amplitudes, measure the angular frequencies of a pendulum.

- A Vernier rotary motion sensor\[7\] will be used to collect data from an oscillating pendulum. The data are to be analyzed using Logger Pro. Open the file: “LoggerPro_Pendulum.cmbl” (available from Blackboard), to begin your measurement.

- Fit your data with a sine function using the “Analyze” → “Curve Fit...” menu option. One of the built-in fitting functions that Logger Pro offers is a sine function of the form \(A \sin(B \times x + C) + D\). Based on the parameters obtained from the fit, determine the observed angular frequency.

- Compare your observations with the predicted angular frequency using Equation 3.14 and the acceleration due to gravity *

5 Lab Reflection

Write a brief reflection to document and summarize your lab work.

Your work will be evaluated using the following rubric:

- Data Analysis & Plots (4 points)
  - State what you intend to accomplish with your experiment. What are you trying to measure?
  - Briefly describe how the apparatus will be used to make the measurement. A well-labeled photograph or diagram is an efficient way to explain the details of the experimental setup.
  - Analyze your data using well-formatted plot(s).

*According to the NGS Surface Gravity model, the value of the acceleration due to gravity near the Bloomberg building is \(9.80095 \pm 0.00002 \text{ m/s}^2\). See https://www.ngs.noaa.gov/TOOLS/Gravity/gravcon.html for more information.
Describe the models that were used to make sense of your data.

• Result(s) and Comparison (4 points)
  – Clearly state the final result(s) of your experiment with an associated uncertainty estimate.
  – Report your result with measurement units and the appropriate significant digits.
  – Compare your result to another relevant quantity. For example, you might compare your measurement to an accepted value or another another group's value.
  – Choose the best tools available to make the comparison meaningful.

• Dominant Source of Uncertainty (4 points)
  – Identify and discuss the dominant source(s) of uncertainty in your result.
  – Use error propagation calculations (as appropriate) to support your explanation.

• Experiment Reflection (4 points)
  – Interpret the evidence (plots, results, calculations, error estimates, etc) that you have presented.
  – Explain how your experimental findings relate to the underlying physical principles.
  – Emphasize interesting features of your experiment and/or highlight unanswered questions that you identified in the course of the experiment.
  – Remember that we are interested in the details of your experiment and not vague theoretical statements.

REFERENCES


[6] Ibid. See Unit 22.7 (pp. 249-250).